

Section 11.5 part 2

normal extensions

11.4 - Splitting Fields - Normal Extensions

Def A field extension $K \supset F$ is called normal when it satisfies:
For every irreducible polynomial $p \in F[x] \subset K[x]$,
if p has a root in K , then p splits completely over K .
 into linear factors

One root in K (for an irreducible)
implies all its roots in K .

Th 11.15 A field extension $K \supset F$
is a splitting field of some polynomial if and only if
the extension $K \supset F$ is finite-dimensional and normal.

Pf

① Straightforward part.

Assume that $K \supset F$ is finite-dimensional and normal.

Wanted:

Let u_1, \dots, u_n be a basis of K over F (as a vector space)

K is a
splitting field

Then $K = F(u_1, \dots, u_n)$:

Indeed $K \supseteq F$, $K \ni u_i$, thus $K \supseteq F(u_1, \dots, u_n)$
 $F(u_1, \dots, u_n) \supseteq K$ because every $k \in K$ is a
linear combination $k = c_1 u_1 + \dots + c_n u_n$ $c_i \in F$
thus $k \in F(u_1, \dots, u_n)$

Since $K \supseteq F$ finite-dim'l, K is algebraic over F (Th 11.9)

every u_i is algebraic over F ; let $p_i \in F[x]$ be the minimal polynomial
of u_i

$p_i(u_i) = 0$, $u_i \in K$; since $K \supseteq F$ is normal,
 p_i splits completely - has all roots in K

K is the splitting field of

$$f = p_1 p_2 \cdots p_n$$

$$F(\text{all roots of } f) = K = F(u_1, \dots, u_n) \supseteq F$$

② Assume that $K \supseteq F$ is the splitting field of $f \in F[x]$

$K = F(\text{all roots of } f)$ - finitely many roots, all algebraic
 $= F(u_1, \dots, u_n)$ Thus $K \supseteq F$ is finite dimensional
by Th 11.10

Wanted: $K \supset F$ is normal

Pick arbitrary irreducible polynomial $p \in F[x]$

Let $v \in K$ such that $p(v) = 0$. Wanted: all roots of p belong to K

$$F \subseteq K \subseteq L$$

'the splitting field of p over K

$$p \in F[x] \subset K[x]$$

(we know that L exists)

Wanted: $K = L$

Wanted: $w \in K$

Let $w \in L$ be any root of p , $w \neq v$

By Cor 11.8, $F(v) \cong F(w)$

} take $\sigma =$ identity map
 $\sigma: F \rightarrow F$

$$F(v) \cong F(w) \cong \frac{F[x]}{(p)}$$

Theorem

Th II.5 implies

$$[K:F] = [K(\omega):F]$$

$$\begin{array}{ccc} K^{\sigma^n} & \xrightarrow{\quad \cong \quad} & K(\omega) \\ \cup_1 & \text{Th II.14} & \cup_1 \\ F(\nu) & \cong & F(\omega) \\ \cup_1 & \text{Cor II.8} & \cup_1 \\ F & \xrightarrow{\sigma = \text{id}} & F \end{array}$$

Wanted:

$\omega \in K$ means

$$\underline{K(\omega) = K}$$

$$\underbrace{[K(\omega):K]}_{=1}$$

$$\underline{K(\omega) = F(u_1, \dots, u_n)(\omega)} = F(u_1, \dots, u_n, \omega) = \underline{F(\omega)(u_1, \dots, u_n)}$$
$$K = F(u_1, \dots, u_n)$$

$K(\omega)$ is the splitting field of f over $F(\omega)$

K is the splitting field of f over F

at the same time

is the splitting field of f over $F(\nu)$

$$f \in F[x] \subset F(\nu)[x]$$

By Th II.5, $[K:F] = [K(\omega):F]$

$$K(\omega) \supseteq K \supseteq F$$

By Th 11.7

$$[K(\omega) : K][K : F] = [K(\omega) : F]$$

$$\cancel{[K(\omega) : K][K : F]} = \cancel{[K : F]}$$

Finally

$$\underbrace{[K(\omega) : K]}_{} = 1$$